Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

EXAMPLE 1 Using Identities

Find $\sin\theta$ and $\cos\theta$ if $\tan\theta = 5$ and $\cos\theta > 0$.

SOLUTION We could solve this problem by the reference triangle techniques of Section 4.3 (see Example 7 in that section), but we will show an alternate solution here using only identities.

First, we note that $\sec^2\theta = 1 + \tan^2\theta = 1 + 5^2 = 26$, so $\sec\theta = \pm\sqrt{26}$.
Since $\sec\theta = \pm\sqrt{26}$, we have $\cos\theta = 1/\sec\theta = 1/\pm\sqrt{26}$.
But $\cos\theta > 0$, so $\cos\theta = 1/\sqrt{26}$.
Finally,

$$\tan\theta = 5$$
$$\frac{\sin\theta}{\cos\theta} = 5$$
$$\sin\theta = 5\cos\theta = 5\left(\frac{1}{\sqrt{26}}\right) = \frac{5}{\sqrt{26}}.$$

Therefore, $\sin\theta = \dfrac{5}{\sqrt{26}}$ and $\cos\theta = \dfrac{1}{\sqrt{26}}$. *Now try Exercise 1.*

If you find yourself preferring the reference triangle method, that's fine. Remember that combining the power of geometry and algebra to solve problems is one of the themes of this book, and the instinct to do so will serve you well in calculus.

Cofunction Identities

If C is the right angle in right $\triangle ABC$, then angles A and B are complements. Notice what happens if we use the usual triangle ratios to define the six trigonometric functions of angles A and B (Figure 5.2).

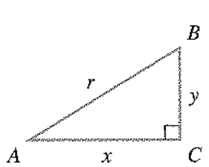


FIGURE 5.2 Angles A and B are complements in right $\triangle ABC$.

Angle A: $\sin A = \dfrac{y}{r}$ $\quad \tan A = \dfrac{y}{x}$ $\quad \sec A = \dfrac{r}{x}$

$\cos A = \dfrac{x}{r}$ $\quad \cot A = \dfrac{x}{y}$ $\quad \csc A = \dfrac{r}{y}$

Angle B: $\sin B = \dfrac{x}{r}$ $\quad \tan B = \dfrac{x}{y}$ $\quad \sec B = \dfrac{r}{y}$

$\cos B = \dfrac{y}{r}$ $\quad \cot B = \dfrac{y}{x}$ $\quad \csc B = \dfrac{r}{x}$

Do you see what happens? In every case, the value of a function at A is the same as the value of its cofunction at B. This always happens with complementary angles; in fact, it is this phenomenon that gives a "co"function its name. The "co" stands for "complement."

Cofunction Identities

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$
$$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$$
$$\sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta \qquad \csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta$$

Although our argument on behalf of these equations was based on acute angles in a triangle, these equations are genuine identities, valid for all real numbers for which both sides of the equation are defined. We could extend our acute-angle argument to produce a general proof, but it will be easier to wait and use the identities of Section 5.3. We will revisit this particular set of fundamental identities in that section.

Odd-Even Identities

We have seen that every basic trigonometric function is either odd or even. Either way, the usual function relationship leads to another fundamental identity.

Odd-Even Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x \qquad \sec(-x) = \sec x \qquad \cot(-x) = -\cot x$$

EXAMPLE 2 Using More Identities

If $\cos\theta = 0.34$, find $\sin(\theta - \pi/2)$.

SOLUTION This problem can best be solved using identities.

$$\sin\left(\theta - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \theta\right) \quad \text{Sine is odd.}$$
$$= -\cos\theta \quad \text{Cofunction identity}$$
$$= -0.34$$

Now try Exercise 7.

Simplifying Trigonometric Expressions

In calculus it is often necessary to deal with expressions that involve trigonometric functions. Some of those expressions start out looking fairly complicated, but it is often possible to use identities along with algebraic techniques (e.g., factoring or combining fractions over a common denominator) to *simplify* the expressions before dealing with them. In some cases the simplifications can be dramatic.

We reject the possibility that $\sin x = 0$ because it would make both sides of the original equation undefined.

The values in the interval $[0, 2\pi)$ that solve $\cos x = 0$ (and therefore $\cos^3 x/\sin x = \cot x$) are $\pi/2$ and $3\pi/2$. *Now try Exercise 51.*

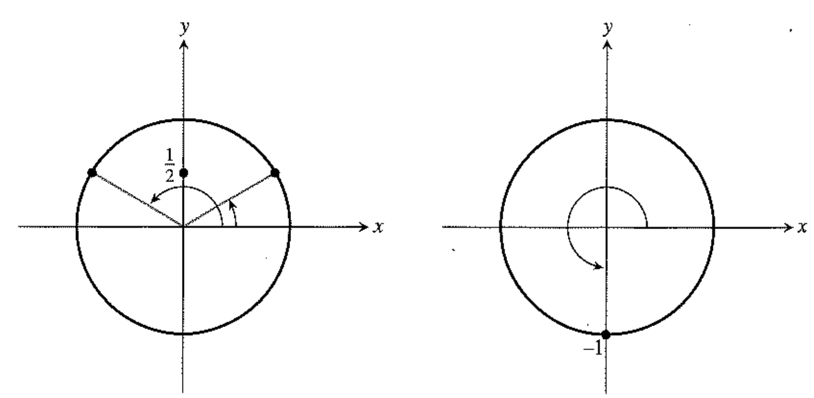
EXAMPLE 7 Solving a Trigonometric Equation by Factoring

Find all solutions to the trigonometric equation $2\sin^2 x + \sin x = 1$.

SOLUTION Let $y = \sin x$. The equation $2y^2 + y = 1$ can be solved by factoring:

$$\begin{aligned} 2y^2 + y &= 1 \\ 2y^2 + y - 1 &= 0 \\ (2y - 1)(y + 1) &= 0 \\ 2y - 1 = 0 \quad &\text{or} \quad y + 1 = 0 \\ y = \frac{1}{2} \quad &\text{or} \quad y = -1 \end{aligned}$$

So, in the original equation, $\sin x = 1/2$ or $\sin x = -1$. Figure 5.5 shows that the solutions in the interval $[0, 2\pi)$ are $\pi/6$, $5\pi/6$, and $3\pi/2$.



(a) (b)

FIGURE 5.5 (a) $\sin x = 1/2$ has two solutions in $[0, 2\pi)$: $\pi/6$ and $5\pi/6$. (b) $\sin x = -1$ has one solution in $[0, 2\pi)$: $3\pi/2$. (Example 7)

To get *all* real solutions, we simply add integer multiples of the period, 2π, of the periodic function $\sin x$:

$$x = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{3\pi}{2} + 2n\pi$$
$$(n = 0, \pm 1, \pm 2, \ldots)$$

Now try Exercise 57.

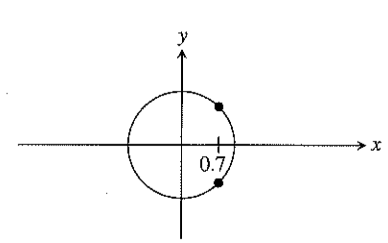


FIGURE 5.6 There are two points on the unit circle with x-coordinate 0.7. (Example 8)

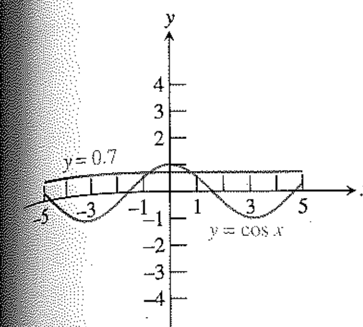


FIGURE 5.7 Intersecting the graphs of $y = \cos x$ and $y = 0.7$ gives two solutions to the equation $\cos t = 0.7$. (Example 8)

You might try solving the equation in Example 7 on your grapher for the sake of comparison. Finding *all* real solutions still requires an understanding of periodicity, and finding *exact* solutions requires the savvy to divide your calculator answers by π. It is likely that anyone who knows that much trigonometry will actually find the algebraic solution to be easier!

EXAMPLE 8 Solving a Trig Equation with a Calculator

Find all solutions to the equation $\cos t = 0.7$, using a calculator where needed.

SOLUTION Figure 5.6 shows that there are two points on the unit circle with an x-coordinate of 0.7. We do not recognize this value as one of our special triangle ratios, but we can use a graphing calculator to find the smallest positive and negative values for which $\cos x = 0.7$ by intersecting the graphs of $y = \cos x$ and $y = 0.7$ (Figure 5.7).

The two values are predictably opposites of each other: $t \approx \pm 0.7954$. Using the period of cosine (which is 2π), we get the complete solution set: $\{\pm 0.7954 + 2n\pi \mid n = 0, \pm 1, \pm 2, \pm 3, \ldots\}$. *Now try Exercise 63.*

QUICK REVIEW 5.1 *(For help, go to Sections A.2, A.3, and 4.7.)*

In Exercises 1–4, evaluate the expression.

1. $\sin^{-1}\left(\frac{12}{13}\right)$
2. $\cos^{-1}\left(\frac{3}{5}\right)$
3. $\cos^{-1}\left(-\frac{4}{5}\right)$
4. $\sin^{-1}\left(-\frac{5}{13}\right)$

In Exercises 5–8, factor the expression into a product of linear factors.

5. $a^2 - 2ab + b^2$
6. $4u^2 + 4u + 1$
7. $2x^2 - 3xy - 2y^2$
8. $2v^2 - 5v - 3$

In Exercises 9–12, simplify the expression.

9. $\frac{1}{x} - \frac{2}{y}$
10. $\frac{a}{x} + \frac{b}{y}$
11. $\frac{x + y}{(1/x) + (1/y)}$
12. $\frac{x}{x - y} - \frac{y}{x + y}$

SECTION 5.1 EXERCISES

In Exercises 1–4, evaluate without using a calculator. Use the Pythagorean identities rather than reference triangles. (See Example 1.)

1. Find $\sin\theta$ and $\cos\theta$ if $\tan\theta = 3/4$ and $\sin\theta > 0$.
2. Find $\sec\theta$ and $\csc\theta$ if $\tan\theta = 3$ and $\cos\theta > 0$.
3. Find $\tan\theta$ and $\cot\theta$ if $\sec\theta = 4$ and $\sin\theta < 0$.
4. Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = 0.8$ and $\tan\theta < 0$.

In Exercises 5–8, use identities to find the value of the expression.

5. If $\sin\theta = 0.45$, find $\cos(\pi/2 - \theta)$.
6. If $\tan(\pi/2 - \theta) = -5.32$, find $\cot\theta$.
7. If $\sin(\theta - \pi/2) = 0.73$, find $\cos(-\theta)$.
8. If $\cot(-\theta) = 7.89$, find $\tan(\theta - \pi/2)$.

In Exercises 9–16, use basic identities to simplify the expression.

9. $\tan x \cos x$
10. $\cot x \tan x$
11. $\sec y \sin(\pi/2 - y)$
12. $\cot u \sin u$
13. $\frac{1 + \tan^2 x}{\csc^2 x}$
14. $\frac{1 - \cos^2\theta}{\sin\theta}$
15. $\cos x - \cos^3 x$
16. $\frac{\sin^2 u + \tan^2 u + \cos^2 u}{\sec u}$

In Exercises 17–22, simplify the expression to either 1 or -1.

17. $\sin x \csc(-x)$
18. $\sec(-x)\cos(-x)$
19. $\cot(-x)\cot(\pi/2 - x)$